An Extension to Three Categories of Belief in a Proposition

As before, there is a finite set of states $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ with $n \geq 2$, and a subset $P \subset \Omega$, with $P \notin \{\emptyset, \Omega\}$, is a proposition. An agent's beliefs $B = \{B_1, B_2\}$ in this extension consist of two disjoint collections of propositions $B_1 = \{P_i\}_{i \in I_1}$ and $B_2 = \{P_i\}_{i \in I_2}$. An information set is an ordered list of propositions $L = (P_1, P_2, ..., P_k)$, and an update rule $U(L)$ associates a belief B to each possible list L. The analogs of sequential processing and stability are:

- **Sequential processing:** For any proposition P and any lists L and L', we have $U_1(L+P) \cup U_2(L+P) \subseteq U_1(L) \cup U_2(L) \cup \{P\}$, and if $U(L) = U(L')$, then $U(L+P) = U(L'+P)$
- **Stability:** For any proposition P' and any list L, if there exists $P \in U_1(L)$ such that $P \subseteq P'$, then $U(L+P') = (U_1(L) \cup \{P'\}, U_2(L))$. Otherwise, if there exists $P \in U_2(L)$ such that $P \subseteq P'$, then either $U(L+P') = (U_1(L) \cup \{P'\}, U_2(L) \setminus \{P'\})$ or $U(L+P') = (U_1(L), U_2(L) \cup \{P'\})$.

  Additionally, if there exists $P \in U_1(L)$ such that $P \in U_2(L+P')$, then we cannot have $P \in U_1(L+P'+P'+...+P')$ for any number of repetitions.

Sequential processing again ensures that belief formation happens in distinct steps after processing each proposition, and the update at each step depends only on the interim belief. The stability axiom has changed in a couple of ways. There are two categories of beliefs now, and propositions in B_1 are more strongly believed than those in B_2. If something in B_1 already implies a new proposition, then it must be put in the highest category. If something in B_2 implies the new proposition P', then it could go in either category—note this allows a repetition of P to move P from B_2 to B_1. The second part of the axiom says that if hearing P' leads to more doubt about P, then hearing P' again cannot lead to less doubt about P.

I impose the following consistency requirements on the set $\mathcal{B}$ of permissible beliefs:

- There exists some state consistent with all propositions in B_1 and B_2: $\bigcap_{i \in I_1 \cup I_2} P_i \neq \emptyset$
- There is no $P \in B_1$ and $P' \in B_2$ such that $P' \subseteq P$.

The first condition simply says the agent can detect contradictions in her beliefs, while the second implies the agent cannot hold a weaker belief in a proposition that is logically implied by one in the highest category.

There are natural analogs of order independence, label neutrality, and openness in this setting. As before, given a permutation π of Ω, I write $\pi(P) = \{\pi(\omega_1), \pi(\omega_2), ..., \pi(\omega_\ell)\}$ for $P = \{\omega_1, \omega_2, ..., \omega_\ell\}$, and I write $L^\pi = (\pi(P_1), \pi(P_2), ..., \pi(P_k))$ for a list $L = (P_1, P_2, ..., P_k)$. I similarly write $B^\pi = \{B_1^\pi, B_2^\pi\}$ for the corresponding permuted beliefs.

- **Order Independence:** If π is a permutation of a list L, then $U(\pi(L)) = U(L)$
- **Label Neutrality:** For any list L, if π is a permutation of Ω, the $U(L^\pi) = U(L)^\pi$

- **Openness:** For any L and any proposition P, there exists L' such that $P \in U_1(L+L')$.

The definitions of order independence and label neutrality have not changed. The definition of openness means that it is always possible to get an arbitrary proposition P into the highest belief category. In the following proposition, the "skeptic rule" refers to the update rule such that $U_1(L) = U_2(L) = \emptyset$ for any list L.

Proposition 1. *There is no update rule that satisfies both order independence and openness.*

The update rule U satisfies label neutrality and order independence if and only if U is the skeptic rule.

Proof. For the first claim, assume for the sake of contradiction that $U(L)$ satisfies openness and order independence. Let ω and ω' be distinct states, and let $\overline{L}$ denote a list containing 3^n copies of all possible propositions. By openness, there exists a list L_ω such that $\{\omega\} \in U(\overline{L} + L_\omega)$, and likewise there exists a list $L_{\omega'}$ such that $\{\omega'\} \in U(\overline{L} + L_{\omega'})$. Consider a permutation of $\overline{L} + L_\omega$ such that all duplicate propositions appear consecutively. I claim that we can remove the propositions in L_ω without changing the resulting beliefs.

When a proposition is repeated multiple times, one of three things can happen eventually. First, the proposition could enter category 1, in which case beliefs stop changing. Second, the proposition could enter category 2, and repetition fails to elevate it to category 1, so beliefs stop changing. Finally, if the agent never accepts the new proposition, there are only so many other propositions that can get demoted to a lower category or promoted to a higher one—eventually, beliefs stop changing. Since $\overline{L}$ contains 3^n copies of every proposition, we are sure to arrive at one of these outcomes before processing repetitions due to the propositions in L_ω. I conclude by order independence that $U(\overline{L} + L_\omega) = U(\overline{L}) = U(\overline{L} + L_{\omega'})$, but we cannot have both $\{\omega\} \in U_1(\overline{L})$ and $\{\omega'\} \in U_1(\overline{L})$.

For the second claim, it should be clear the skeptic rule satisfies order independence and label neutrality. Suppose there exists a proposition P such that $P \in U_i(P)$ for some $i \in \{1, 2\}$. By label neutrality, we have $P' \in U_i(P')$ for any P' such that $|P'| = |P|$. Let L denote a list containing two copies of all such P'. Label neutrality and order independence imply that for any permutation π of Ω, we have

$$U^\pi(L) = U(L^\pi) = U(L).$$

This implies that $U_1(L) = U_2(L) = \emptyset$. Sequential processing now implies that $P \in U_i(L+P)$, but order inependence and stability imply that $U_i(L + P) = \emptyset$—just move the three copies of P to the front of the list and note the third repetition has no effect, so it can be removed. Hence, the skeptic rule is the only rule satisfying label neutrality and order independence. □

I note that the conclusion of the second part is sensitive to the set of permissible beliefs. If we exclude contradictory beliefs from B_1, but allow contradictory propositions in the weaker category B_2, then we get an analogous result that any update rule satisfying order independence and label neutrality always has $U_1(L) = \emptyset$, but it may allow propositions into the weaker category of belief.