Economics of Networks
Introduction to Game Theory: Part 2

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Agenda

Recap of last time

Mixed strategies and mixed strategy equilibrium

Existence of Nash Equilibria

Extensive form games and subgame perfection

Reading: Osborne chapters 4-6

Recap

Rational choice:

- Agents described by preferences, can represent as utility function
- With uncertainty, maximize expected utility

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Dominant and dominated strategies

- Intuitive game solutions
- Can't always get a unique prediction

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- Intuitive game solutions
- Can't always get a unique prediction

Pure strategy Nash Equilibrium

- Everyone plays a best response
- Doesn't always exist...

Nonexistence

Recall the matching pennies game:

	Heads	Tails
Heads	$(-1, 1)$	$(1, -1)$
Tails	(1,-1)	$(-1, 1)$

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How would you play?

Nonexistence

Alternative interpretation: Penalty Kicker and Goalie

Kicker / Goalie	Left	Right
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Is it a good strategy for the kicker to always kick to the left side of the net?

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Is it a good strategy for the kicker to always kick to the left side of the net?

Empirical evidence suggests that most penalty kickers "randomize"

- Mixed strategies

Mixed Strategies

Let Σ_i denote the set of all lotteries over pure strategies in S_i

- In our example, a mixed strategy is a probability of kicking (or diving) left

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Payoff is expected utility:

$$u_i(\sigma) = \int_S u_i(s) \mathrm{d}\sigma(s)$$

Mixed Strategy Nash Equilibrium

Definition

A mixed strategy profile σ^* is a Nash Equilibrium if for each player i and all $\sigma_i \in \Sigma_i$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

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- Best response to correct conjecture

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Space of lotteries is large, how do we tell we have an equilibrium?

Mixed Strategy Nash Equilibrium

Proposition

In a normal form game, the profile $\sigma^* \in \Sigma$ is a Nash Equilibrium if and only if for each player i, every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*.

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Proof idea: If we put positive probability on a strategy that is not a best response, shifting that probability to a best response strictly increases utility.

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Consequence: every action in support of i's equilibrium mixed strategy yields same expected payoff

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Matching pennies: unique mixed Nash equilibrium, players put probability $\frac{1}{2}$ on heads

	Heads	Tails
Heads	$(-1, 1)$	$(1, -1)$
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Example: Work or Shirk

Recall the partnership game:

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- Are there mixed equilibria?

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Two pure strategy equilibria

- Are there mixed equilibria?

Yes! Both randomize with probability $\frac{1}{2}$

- Expected payoff of $\frac{1}{2}$

Interpretation of Mixed Equilibria

Deliberate choice to randomize

- Recall our penalty kicker
- Bluffing in poker

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Steady state of a learning process

Distribution of outcomes in a perturbed game with pure strategy best responses

- "Purification"

Nash's Theorem

Theorem

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Why do we care?

- Without existence, studying properties of equilibria is difficult (maybe meaningless)
- Knowing existence, we can just try to find the equilibria

Tools: Weierstrass's Theorem

Theorem (Weierstrass)

Let A be a nonempty compact subset of a finite dimensional Euclidean space, and let $f : A \to \mathbb{R}$ be a continuous function. The function f attains a maximum and a minimum in A.

Tools: Weierstrass's Theorem

Theorem (Weierstrass)

Let A be a nonempty compact subset of a finite dimensional Euclidean space, and let $f : A \to \mathbb{R}$ be a continuous function. The function f attains a maximum and a minimum in A.

Recall definition of compactness: every sequence has a convergent subsequence

- Continuity ensures sup and inf are contained in the image $f(A)$

Tools: Kakutani's Fixed Point Theorem

Theorem (Kakutani)

Let $f : A \rightrightarrows A$ be a correspondence, i.e. $x \in A \implies f(x) \subset A$, satisfying:

- *A is a non-empty compact and convex subset of a finite dimensional Euclidean space*
- *$f(x)$ is non-empty for all $x \in A$*
- *$f(x)$ is convex valued*
- *$f(x)$ has a closed graph, i.e. $(x_n, y_n) \to (x, y)$ with $y_n \in f(x_n)$ implies $y \in f(x)$*

Then f has a fixed point: there exists $x \in A$ such that $x \in f(x)$

Definitions

A set in Euclidean space is compact iff it is bounded and closed

- Every infinite sequence has a convergent subsequence

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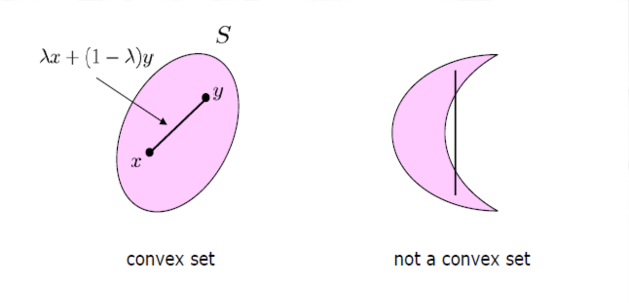
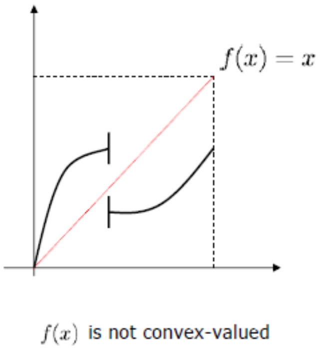
A set S is convex if for any $x, y \in S$ and any $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in S$.



Kakutani's Fixed Point Theorem, Illustration



Proof of Nash's Theorem

Recall σ^* is a mixed strategy Nash Equilibrium if for every player i and every $\sigma_i \in \Sigma_i$,

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Define best response correspondence $B_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ for player i:

$$B_i(\sigma_{-i}) = \{\sigma_i' \in \Sigma_i \,:\, u_i(\sigma_i', \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i}),\, \forall\, \hat{\sigma}_i \in \Sigma_i\}$$

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Set of best response correspondences

$$B(\sigma) = \{B_i(\sigma_{-i})\}_{i \in N}$$

Proof, continued

We apply Kakutani's fixed point theorem to the correspondence $B : \Sigma \rightrightarrows \Sigma$

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Need to show:

- Σ is compact, convex, and non-empty
- $B(\sigma)$ is non-empty
- $B(\sigma)$ is convex-valued
- $B(\sigma)$ has a closed graph

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$\Sigma = \prod_{i \in N} \Sigma_i$ is compact, convex, and non-empty by definition

- Σ_i is a simplex of dimension $|S_i| - 1$

Proof, continued

$B(\sigma)$ is non-empty by Weierstrass's theorem

- Σ_i is non-empty and compact, so u_i attains its maximum for each i

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For any $\hat{\sigma}_i$, we have

$$\begin{aligned} u_i\left(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}\right) &= \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i}) \\ &\geq \lambda u_i(\hat{\sigma}_i, \sigma_{-i}) + (1-\lambda)u_i(\hat{\sigma}_i, \sigma_{-i}) \\ &= u_i(\hat{\sigma}_i, \sigma_{-i}) \end{aligned}$$

Proof, continued

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- That is, there exists i such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$
- Since $\hat{\sigma} \notin B_i(\sigma_{-i})$, there exists $\sigma_i' \in \Sigma_i$ and $\epsilon > 0$ such that

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$$

Proof, continued

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- By continuity, for sufficiently large n we have

$$u_i(\sigma_i', \sigma_{-i}^n) \geq u_i(\sigma_i', \sigma_{-i}) - \epsilon$$

Proof, continued

Combining the last two inequalities, we have

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon$$

Proof, continued

Combining the last two inequalities, we have

$$u_i(\sigma_i', \sigma_{-i}^n) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \epsilon$$

This contradicts assumption that $\hat{\sigma}_i^n \in B_i(\sigma_{-i}^n)$

- We conclude that B has a closed graph

Proof, continued

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By definition, σ^* is a mixed strategy equilibrium if $\sigma^* \in B(\sigma^*)$

Equilibrium existence follows from Kakutani's theorem

Equilibrium Existence in Infinite Games

A similar theorem gives existence of pure strategy equilibria in infinite games

Theorem (Debreu, Glicksburg, Fan)

Consider an infinite normal form game $(N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ such that for each $i \in N$:

- S_i *is compact and convex*
- $u_i(s_i, s_{-i})$ *is continuous in* s_{-i}
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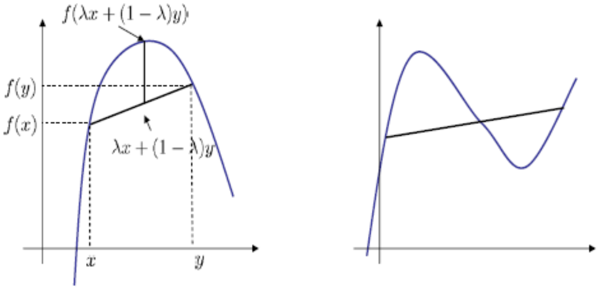
Then a pure strategy Nash Equilibrium exists.

Proof left as exercise

Definitions

Suppose S is a convex set. Then a function $f : S \to \mathbb{R}$ is concave if for any $x, y \in S$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$



concave function   not a concave function

More Existence Questions

Can we relax concavity?

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Example:

- Two players simultaneously pick locations $s_1, s_2 \in \mathbb{R}^2$ on the unit circle
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There are mixed strategy equilibria...

A Stronger Theorem

Theorem (Glicksberg)

Consider an infinite normal form game such that

- S_i *is compact and convex for each* i
- $u_i(s_i, s_{-i})$ *is continuous in both arguments*

Then a mixed strategy Nash Equilibrium exists.

Proof is beyond scope of this class

Extensive Form Games

Up to now, we have ignored dynamics

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Extensive form games capture strategic situations with multiple actions in sequence

- For now, focus on games with observable actions

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Represent extensive form using a game tree

- Keep track of possible histories

Definitions

Extensive form game is a collection $(N, H, Z, \{A_i^h\}_{\substack{i \in N \\ h \in H}}, \{u_i\}_{i \in N})$

- Set of players N
- Set of non-terminal histories H
- Set of terminal histories Z
- Actions A_i^h for each player i at each non-terminal history h
- Payoff function u_i giving payoff to i at each terminal history

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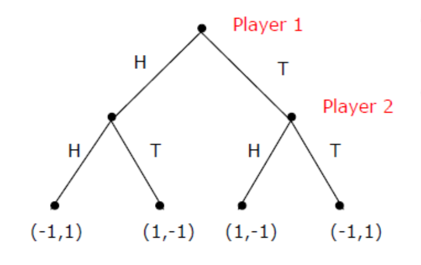

Strategies in Extensive Form Games

A strategy for player i is a map s_i giving an action for each non-terminal history $h \in H$

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In example, player 1 has two strategies, H and T

How many does player 2 have?

- Four: HH, HT, TH, TT

Strategies in Extensive Form Games

Can use strategies to express extensive form game in normal form

- Action in normal form game is choice of a complete contingent plan

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Normal form of two-stage matching pennies:

Player 1 / Player 2	HH	HT	TH	TT
Heads	$(-1, 1)$	$(-1, 1)$	$(1, -1)$	$(1, -1)$
Tails	$(1, -1)$	$(-1, 1)$	$(1, -1)$	$(-1, 1)$

Sidebar: Normal Form to Extensive Form

Recall the original matching pennies example: players choose heads/tails simultaneously

Sidebar: Normal Form to Extensive Form

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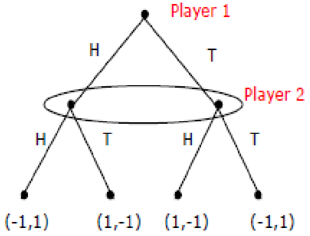
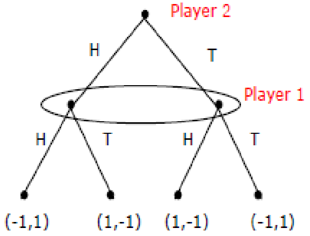
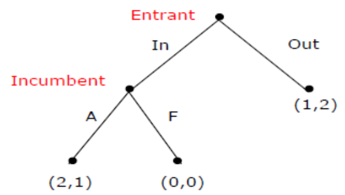
Can represent using a game tree by adding information sets

- Player cannot distinguish two decision nodes in same information set





Example: Entry Deterrence



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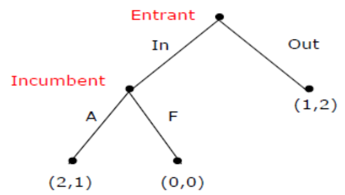

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Entrant / Incumbent	Accommodate	Fight
In	$(2, 1)$	$(0, 0)$
Out	$(1, 2)$	$(1, 2)$

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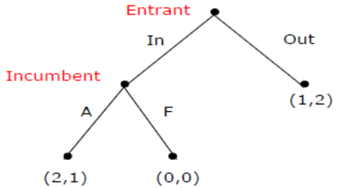

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Two pure Nash equilibria: (In, A) and (Out, F).

Are Both Equilibria Reasonable?

Equilibrium (Out, F) sustained by noncredible threat

- After observing entry, best response is to accommodate

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Refinement by "subgame perfection"

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Refinement by "subgame perfection"

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Need to formally define a subgame

Subgame Perfect Equilibrium

Recall an extensive form game is expressed as a game tree

- Let V_G denote the set of nodes

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An information set $X \subseteq V_G$ is a successor of node y (written $X \succ y$) if we can reach X through y

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An information set $X \subseteq V_G$ is a successor of node y (written $X \succ y$) if we can reach X through y

Definition

Let x denote a node and let G_x denote x and all its successors. The set G_x is a subgame if

- Node x is alone in its information set
- If $y \in G_x$ is a successor of $z \notin G_x$, then $x \succ z$

Subgame Perfect Equilibrium

A restriction of a strategy profile σ to the subgame G_x, written $\sigma_{|G_x}$ is the profile implied by σ in the subgame G_x

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Rules out non-credible threats

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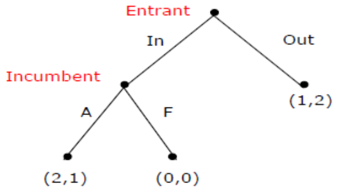
How to find subgame perfect equilibria?

Backward Induction

Backward induction: start from the last subgames, find Nash equilibria of those, then work backwards towards the beginning of the game



Existence of Subgame Perfect Equilibria

Theorem

Every finite perfect information extensive form game G has a pure strategy SPE

Note: perfect information means all information sets contain exactly one node

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Theorem

Every finite extensive form game G has a SPE

Follow's from Nash's theorem

Value of Commitment

What if the incumbent firm could commit to fight?

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Consider a dynamic version of Cournot competition

- Firm 1 commits to a quantity of output first
- Only after this does firm 2 choose a quantity

Stackleberg Competition

Recall the two firms will face a market price $p = 1 - q_1 - q_2$

- Firm i earns $q_i(p - c)$

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Firm 1 chooses q_1 to maximize

$$q_1(p - c) = q_1\left(1 - q_1 - \frac{1 - q_1 - c}{2} - c\right) = q_1\left(\frac{1 - q_1 - c}{2}\right)$$

giving $q_1 = \frac{1-c}{2}$.

- Total output is higher in the Stackleberg equilibrium (why?)

Recap

Nash equilibrium will be our workhorse solution concept

- Can essentially always guarantee existence of a (mixed strategy) equilibrium

Will employ refinements, especially in dynamic games, where appropriate

Next time: a network application of basic game theory

- Traffic routing