Economics of Networks
Social Learning

Evan Sadler

Massachusetts Institute of Technology

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Agenda

- Recap of rational herding
- Observational learning in a network
- DeGroot learning

Reading: Golub and Sadler (2016), “Learning in Social Networks”

Supplement: Acemoglu et al., (2011), “Bayesian Learning in Social Networks;” Golub and Jackson (2010), “Naïve Learning in Social Networks and the Wisdom of Crowds”

The Classic Herding Model

Two equally likely states of the world $\theta \in \{0, 1\}$

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Signals i.i.d. conditional on the state:

$$\mathbb{P}(s_n = 0 \,|\, \theta = 0) = \mathbb{P}(s_n = 1 \,|\, \theta = 1) = g > \frac{1}{2}$$

Rational Herding

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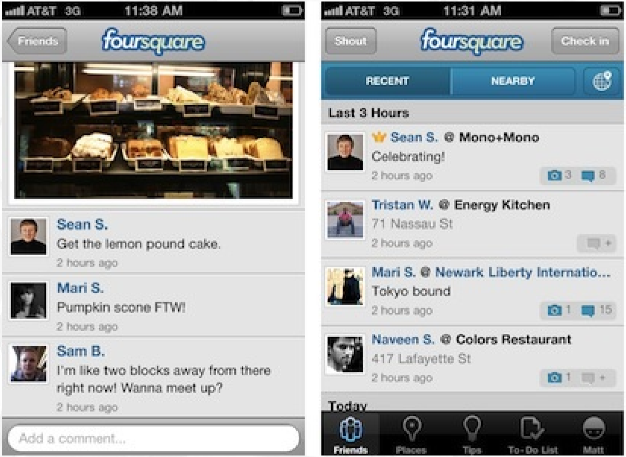
Inefficiency reflects an informational externality

- Agents fail to internalize the value of their information to others

Observational Learning: A Modern Perspective



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How does the observation structure affect learning?

A Souped-up Model

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Signals are conditionally i.i.d. with distributions $\mathbb{F}_\theta$

The Observation Structure

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- $\mathbb{Q}$ is common knowledge
- For this class, assume $\{B(n)\}_{n \in \mathbb{N}}$ are mutually independent

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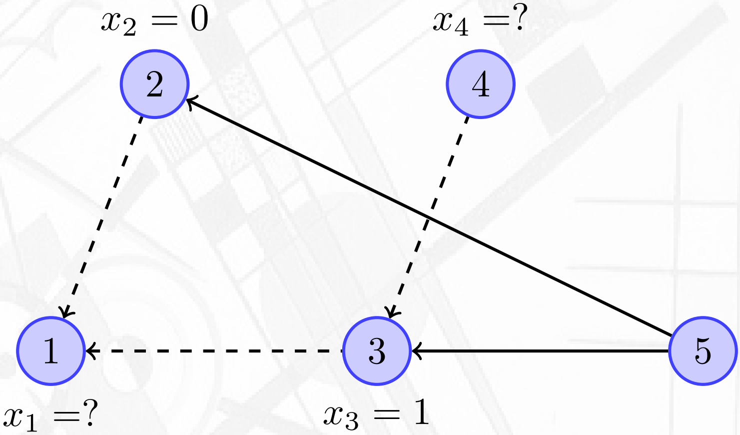
Study perfect Bayesian equilibria σ of the learning game:

$$\sigma_n = \arg\max \mathbb{E}_\sigma \left[u(x, \theta) \,|\, \mathcal{I}_n \right]$$

A Complex Inference Problem



Learning Principles

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Corresponding learning metrics: diffusion vs. aggregation

Private and Social Beliefs

Define the private belief $p_n = \mathbb{P}(\theta = 1 \mid s_n)$, distribution $\mathbb{G}_\theta$

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Support of private beliefs $[\underline{\beta}, \overline{\beta}]$

$$\underline{\beta} = \inf\{r \in [0,1] \,:\, \mathbb{P}(p_1 \leq r) > 0\}$$

$$\overline{\beta} = \sup\{r \in [0,1] \,:\, \mathbb{P}(p_1 \leq r) < 1\}$$

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The expert signal $\tilde{s}$, binary with

$$\mathbb{P}(\theta = 1 \mid \tilde{s} = 0) = \underline{\beta}, \quad \mathbb{P}(\theta = 1 \mid \tilde{s} = 1) = \overline{\beta}$$

Learning Metrics

Information diffuses if

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A network topology $\mathbb{Q}$ diffuses (aggregates) information if diffusion (aggregation) occurs for *every* signal structure and *every* equilibrium strategy profile

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Our definition emphasizes role of network

- Complete network diffuses, does not aggregate, information

Necessary Conditions for Learning

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Theorem

If $\mathbb{Q}$ diffuses information, we must have expanding subnetworks:

$$\lim_{n \to \infty} \mathbb{P}(|\hat{B}(n)| < K) = 0$$

for all $K \in \mathbb{N}$

The Improvement Principle

Intuition: I can always pick a neighbor to copy

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- Key idea: imitate this neighbor if my signal is weak, follow my signal if it is strong

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Suboptimal rule, but it gives a lower bound on performance

- Rational agents must do (weakly) better

Two Lemmas

Lemma

Suppose $\mathbb{Q}$ *has expanding subnetworks, and there exists a continuous increasing* $\mathcal{Z}$ *such that* $\mathcal{Z}(u) > u$ *for all* $u < u^*$*, and*

$$\mathbb{E}_\sigma[u(x_n, \theta)] \geq \mathcal{Z}(\mathbb{E}_\sigma[u(x_{\overline{B}(n)}, \theta)])$$

Then $\mathbb{Q}$ *diffuses information.*

Lemma

There exists a continuous increasing $\mathcal{Z}$ *with* $\mathcal{Z}(u) > u$ *for all* $u < u^*$ *such that*

$$\mathbb{E}_\sigma[u(x_n, \theta)] \geq \mathcal{Z}(\mathbb{E}_\sigma[u(x_m, \theta)])$$

for any $m \in B(n)$*.*

A Key Assumption: Independent Neighborhoods

Our two lemmas imply that information diffuses in any sufficiently connected network

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If neighborhoods are correlated, the fact that I observe someone is related to how informative their choice is

Failure to Aggregate

Proposition (Acemoglu et al., 2011, Theorem 3)

The topology $\mathbb{Q}$ fails to aggregate information if any of the following conditions hold:

- $B(n) = \{1, 2, ..., n-1\}$
- $|B(n)| \leq 1$ *for all* n
- $|B(n)| \leq M$ *for all* n *and some* $M \in \mathbb{N}$, *and*

$$\lim_{n \to \infty} \max_{m \in B(n)} m = \infty \quad \text{almost surely}$$

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Proposition

Suppose there exists a subsequence $\{m_i\}$ *such that*

$$\sum_{i \in \mathbb{N}} \mathbb{P}(B(m_i) = \emptyset) = \infty, \text{ and } \lim_{n \to \infty} \mathbb{P}(m_i \in B(n)) = 1$$

for all i. *Then* $\mathbb{Q}$ *aggregates information.*

Follows from a martingale convergence argument

Heterogeneous Preferences

Key limitation so far: everyone has the same preferences

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Payoffs

$$u(x, \theta, t) = \begin{cases} 1 - \theta + t & \text{if } x = 0 \\ \theta + 1 - t & \text{if } x = 1 \end{cases}$$

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The type t parameterizes the relative cost of error in each state

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Assume

- $B(n) = \{n-1\}$ for all n
- Odds have type $\frac{1}{5}$, evens have type $\frac{4}{5}$
- $\mathbb{G}_0(r) = 2r - r^2$ and $\mathbb{G}_1(r) = r^2$

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Can show inductively that all odds (evens) err in state 0 (state 1) with probability at least $\frac{1}{4}$ (homework problem)

Robust Large-Sample Principle

With full support in preference distribution, preferences can counterbalance social information

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Proposition

Suppose preference types are i.i.d. with full support on $(0, 1)$*, and there exists an infinite sequence* $\{m_i\}$ *such that*

$$\lim_{n \to \infty} \mathbb{P}(m_i \in B(n)) = 1$$

for all i*. Then information aggregates.*

Remarks on the SSLM

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Can't say much about rate of learning, influence

A Different Approach

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- How likely is it the state is 1?
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A simple update rule:

$$x_i(t) = \sum_{j \in N} W_{ij} x_j(t-1)$$

Think of W as a weighted graph

DeGroot Updating

Assumptions:

- The $x_i(0)$ are given exogenously
- The matrix W is an $n \times n$ matrix with non-negative entries
- For each i we have $\sum_{j \in N} W_{ij} = 1$

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Simple example:

- Consider an unweighted graph G, agent i has degree d_i
- $W_{ij} = \frac{1}{d_i}$ for each neighbor j of i, and $W_{ij} = 0$ for each non-neighbor

Matrix Powers and Markov Chains

Can rewrite the update rule as

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How to think about W_{ij}^t

- $\frac{\partial x_i(t)}{\partial x_j(0)} = W_{ij}^t$: influence of j on i's time t opinion
- W_{ij}^t sums over all paths of indirect influence

The Long-Run Limit

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Start with strongly connected networks

- The network W is strongly connected if there is a directed path from i to j for every $i, j \in N$

Call W *primitive* if there exists q such that every entry of W^q is strictly positive

- Equivalent to aperiodicity in the network

The Long-Run Limit

Theorem

Suppose W *is strongly connected and aperiodic. The limit* $\lim_{t\to\infty} x_i(t)$ *exists and is the same for each* i.

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Theorem

Suppose W is strongly connected and aperiodic. The limit $\lim_{t\to\infty} x_i(t)$ exists and is the same for each i.

Proof:

- The sequence $\max_i x_i(t)$ is monotonically decreasing
- The sequence $\min_i x_i(t)$ is monotonically increasing
- Primitivity ensures the two extreme agents put at least weight $w > 0$ on each other after q steps
- Distance between max and min decreases by factor at least $1 - w$ after every q steps

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Vector $\boldsymbol{\pi}$ must satisfy

$$\boldsymbol{\pi}^T W = \boldsymbol{\pi}$$

Left eigenvector with eigenvalue 1

Influence on the Consensus

Theorem

If W is strongly connected and primitive, then for all i

$$\lim_{t\to\infty} x_i(t) = \sum_{i\in N} \pi_i x_i(0)$$

where π_i is the left eigenvector centrality of i in W

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Note vector $\boldsymbol{\pi}$ is also the unique stationary distribution of the Markov chain with transition probabilities given by W

Can also be seen as a consequence of the Perron-Frobenius Theorem from linear algebra

Beyond Strong Connectedness

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Small amount of communcation across classes makes large (discontinuous) difference in asymptotic outcomes

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Consider an infinite sequence of networks $\{W^{(n)}\}_{n=1}^{\infty}$, population getting larger

If $x^{(n)}(\infty)$ is the consensus estimate in network n, do these estimates converge to μ as $n \to \infty$?

When is Consensus Correct?

Theorem (Golub and Jackson, 2010)

The consensus beliefs $x^{(n)}(\infty)$ converge in probability to μ if and only if

$$\lim_{n\to\infty} \max_i \pi_i^{(n)} = 0.$$

The influence of the most central agent in the network converges to zero

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Proof:

- We have $Var\left[x^{(n)}(\infty) - \mu\right] = \sum_{i=1}^{n}(\pi_i^{(n)})^2\sigma^2$
- Converges to zero if and only if $\max_i \pi_i^{(n)} \to 0$
- If not, no convergence in probability
- If it does, Chebyshev's inequality implies convergence in probability

Speed of Convergence

Consensus might be irrelevant if it takes too long to get there

- How long does it take for differences to get “small”?
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Focus on worst-case convergence time, highlight role of network

A Spectral Decomposition

Lemma

For "generic" W, we may write

$$W^t = \sum_{l=1}^{n} \lambda_l^t P_l$$

where

- $1 = \lambda_1, \lambda_2, ..., \lambda_n$ *are* n *distinct eigenvalues of* W
- P_l *is a projection onto the eigenspace of* λ_l
- $P_1 = W^\infty$ *and* $P_1 \mathbf{x}(0) = \mathbf{x}(\infty)$
- $P_l \mathbf{1} = 0$ *for all* $l > 1$, *where* $\mathbf{1}$ *is a vector of all ones*

All other eigenvalues strictly smaller in absolute value than $\lambda_1 = 1$

Speed of Convergence

Theorem

For generic W,

$$\frac{1}{2}|\lambda_2|^t - (n-2)|\lambda_3|^t \leq \sup_{\mathbf{x}(0)\in[0,1]^n} \|\mathbf{x}(t) - \mathbf{x}(\infty)\|_\infty \leq (n-1)|\lambda_2|^t.$$

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Note $\|\cdot\|_\infty$ denotes the supremum norm, largest deviation from consensus among all agents

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Clear answer to first question: rate of convergence depends on second largest eigenvalue

- Larger λ_2 (i.e. smaller spectral gap) implies slower convergence

Segregation and Slow Convergence

What network features correspond to large $|\lambda_2|$?

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Define the *bottleneck ratio*

$$\Phi(W) = \min_{\substack{M \subseteq N \\ \pi(M) \geq \frac{1}{2}}} \frac{\sum_{i \in M, j \notin M} \pi_i W_{ij}}{\sum_{i \in M} \pi_i}$$

Small when some influential group pays little attention to those outside itself

- Can use to bound size of $|\lambda_2|$

Wrap Up

Limited ability to learn through observation

- Information externality creates inefficiency
- Heterogeneity may help or hurt depending on network properties

Naïve learning model gives measures of influence, learning rate

Next time: moving on to models of diffusion, different influence mechanism